Inference for Linear Regression

ANOVA for SLR

Grinnell College

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Review

- Hypothesis testing
  - test-statistics
  - p-values
  - need to be careful what H_0 and H_A actually are

- ANOVA
  - testing equality of group means
  - H_0: $\mu_1 = \mu_2 = \cdots = \mu_k$
  - $\mathrm{F} = \frac{MSG}{MSE} = \frac{SSG/(k-1)}{SSE/(n-k)}$
  - MSG measures how far (on average) group means are from overall mean
  - MSE measures how far (on average) observations are from their group means

ANOVA and Regression

ANOVA Null hypothesis:

$$H_0 : \mu_1 = \mu_2 = \dots \mu_k$$

- comparing mean values of a continuous variable for k different groups
- H_0 true $\implies$ each group has same *overall* mean μ

We are going to see how this ANOVA stuff can be applied to linear regression

ANOVA and Regression

We might ask if it is better to predict an outcome ($\hat{y}$) using an overall mean or if we are better off predicting with a group mean:

$$H_0 : \hat{y}_j = \mu, \qquad H_A : \hat{y}_j = \mu_j$$

In this case by *better*, we mean that we minimize the residual sum of squares, or the squared difference between our prediction and the true value

$$\begin{aligned} \text{Sums of Squared Residuals} &= \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \\ &= \sum_{i=1}^{n} e_i^2 \end{aligned}$$

Regression

Recall that regression formulas are of the form:

$$y_i = \beta_0 + X_i\beta_1 + \epsilon_i$$

- β_0 represents an intercept
- β_1 indicates a slope associated with X_i

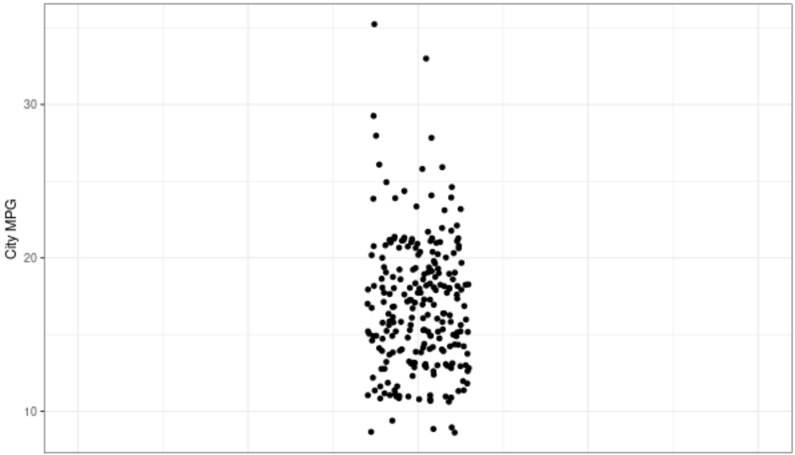
Once we fit to line to the data, we have an estimated line of

$$\hat{y}_i = \hat{\beta}_0 + X_i\hat{\beta}_1 \ \ (= b_0 + b_1 X_i)$$

- residual $e_i = y_i - \hat{y}_i$ is an estimate of the error ϵ_i

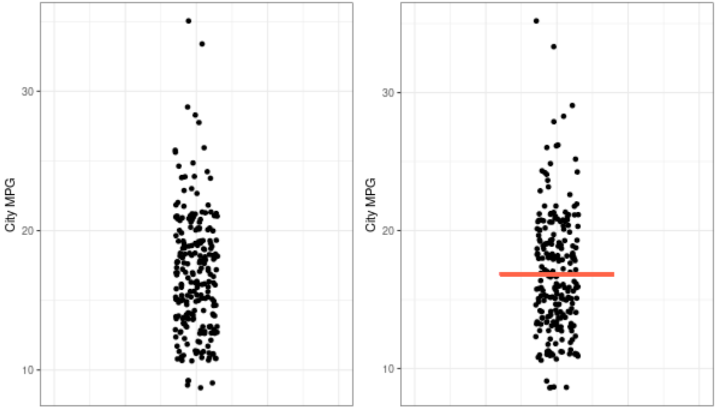
mpg Example

Consider the mpg dataset, where we might be interested in estimating the city miles per gallon of various vehicles



mpg Example

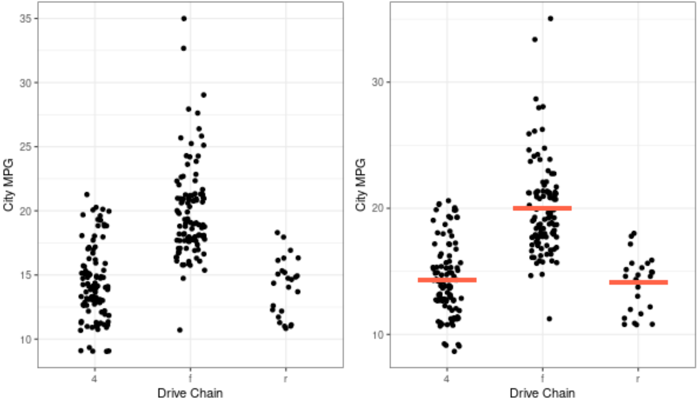
Using simply the overall mean, we would have total squared error of 4220



	Df	Sum Sq	Mean Sq	F value	Pr(>F)
Residuals	233	4220.35	18.11		

mpg Example

Consider the alternative, where we predict city mileage based on drive train



	Df	Sum Sq	Mean Sq	F value	Pr(>F)
drv	2	1878.81	939.41	92.68	<0.0001
Residuals	231	2341.53	10.14		

- SSR has gone down (good!) and is sequestered into SSG (drv)

mpg Example

In terms of a regression model, we could frame this as

$$\hat{y} = \mathbb{1}_{\text{4wd}}\hat{\beta}_1 + \mathbb{1}_{\text{Fwd}}\hat{\beta}_2 + \mathbb{1}_{\text{Rwd}}\hat{\beta}_3$$

where $\mathbb{1}$ represents our *indicator variable* and, in the case of categorical variable regression, $\hat{\beta}$ represents the mean value for each group. This is exactly what we saw towards the beginning of the semester

```
> lm(cty ~ -1 + drv, mpg)

Coefficients:
 drv4   drvf   drvr
14.33  19.97  14.08
```

$$\hat{y} = (14.33 \times \mathbb{1}_{\text{4wd}}) + (19.97 \times \mathbb{1}_{\text{Fwd}}) + (14.08 \times \mathbb{1}_{\text{Rwd}})$$

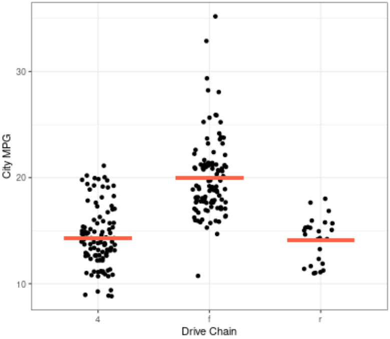
Baseline Category

By default, R will choose one category as the “reference” variable

- usually based on 1st alphabetic category or lowest numeric

```
> lm(cty ~ drv, mpg)
(Intercept)          drvf          drvr
    14.3301        5.6416       -0.2501
```

$$\hat{y} = \hat{\beta}_0 + \mathbb{1}_{\text{Fwd}}\hat{\beta}_1 + \mathbb{1}_{\text{Rwd}}\hat{\beta}_2 = 14.33 + 5.64 \times \mathbb{1}_{\text{Fwd}} - 0.25 \times \mathbb{1}_{\text{Rwd}}$$



Inference and Regression

So, what we have just seen tells us:

- SLR with one categorical variable as a predictor is actually a special case of ANOVA
- both attempted to minimize SSE (=SSR) by partioning that variance into something else (SSG)

However, instead of simply assessing whether or not there is *any* difference between groups, we may be interested specifically in estimating values of β in the expression

$$y = \beta_0 + X\beta_1 + \epsilon$$

where X is a *quantitative* variable

Inference and Regression

$$y = \beta_0 + \beta_1 X + \epsilon$$

When considering a regression line, we are actually trying to find out if there is a linear relationship between the variables.

We could test this by structuring a null hypothesis like so:

$$\begin{aligned} & H_0 : \text{there is no linear relationship} \\ \text{(equivalently)} \quad & H_0 : \beta_1 = 0 \end{aligned}$$

Given our estimate of $\hat{\beta}$, we can make the test statistic,

$$t = \frac{\hat{\beta}_1}{SE_{\beta_1}}$$

mpg Example

Comparing residuals and F statistic for ANOVA and regression

```
> aov(cty ~ drv, mpg) %>% summary()
             Df Sum Sq Mean Sq F value Pr(>F)
drv           2   1879   939.4   92.68 <2e-16 ***
Residuals   231   2342    10.1
```

```
> lm(cty ~ drv, mpg) %>% summary()

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  14.3301     0.3137  45.680   <2e-16 ***
drvf          5.6416     0.4405  12.807   <2e-16 ***
drvr         -0.2501     0.7098  -0.352    0.725


Residual standard error: 3.184 on 231 degrees of freedom
Multiple R-squared:  0.4452,  Adjusted R-squared:  0.4404
F-statistic: 92.68 on 2 and 231 DF,  p-value: < 2.2e-16
```

mpg Example

Comparing pairwise differences for TukeyHSD and regression (reference/intercept variable is 4WD)

```
> aov(cty ~ drv, mpg) %>% TukeyHSD()
  Tukey multiple comparisons of means
    95% family-wise confidence level

            diff        lwr        upr     p adj
f-4  5.6416010  4.602497  6.680705 0.0000000
r-4 -0.2500971 -1.924554  1.424359 0.9338857
r-f -5.8916981 -7.561520 -4.221876 0.0000000
```

```
> lm(cty ~ drv, mpg) %>% summary()

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  14.3301     0.3137  45.680   <2e-16 ***
drvf          5.6416     0.4405  12.807   <2e-16 ***
drvr         -0.2501     0.7098  -0.352    0.725
```

ANOVA and Regression

ANOVA is a generalization of the t-test for multiple groups

- testing: are these groups equal in terms of their means?
- only tells us that a difference exists, not *what* the difference actually is
- hidden assumption for Normal distribution of groups (equiv. residuals for each group)

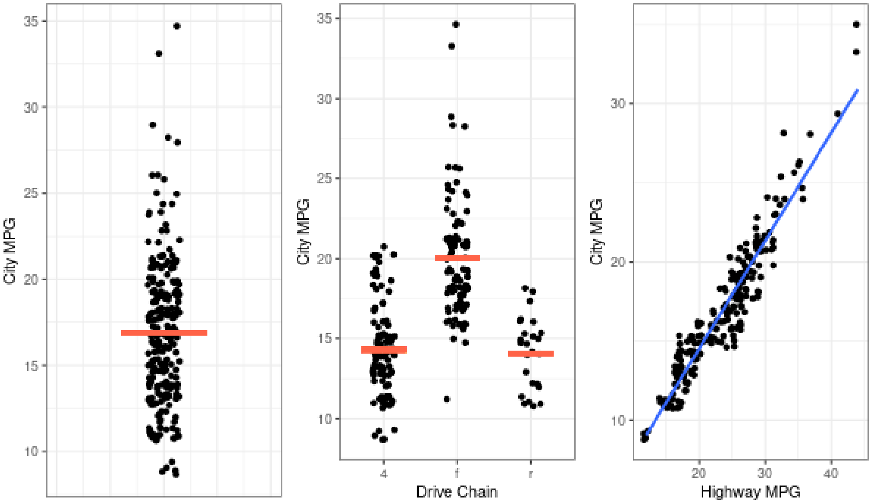
Benefits of Regression:

- provides statistical tests for each of the group categories
- stronger relationship conditions (linear for quant. variables)
- allows us to predict quantitative outcome using a quantitative predictor

Regression Example

Which of these do you suspect will have the smallest residual error?

- think about how far observations are from predictions



mpg Example

```
> lm(cty ~ hwy, mpg) %>% summary()


Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.84420    0.33319   2.534   0.0119 *
hwy          0.68322    0.01378  49.585   <2e-16 ***


Residual standard error: 1.252 on 232 degrees of freedom
Multiple R-squared:  0.9138,  Adjusted R-squared:  0.9134
F-statistic:  2459 on 1 and 232 DF,  p-value: < 2.2e-16
```

$$\hat{y} = b_0 + b_1 \times (hwy) = 0.84 + 0.68 \times (hwy)$$

- F is testing whether both intercept and slope are zero
- t is testing for specifically slope/intercept one at a time
- it is possible that the F-test shows a linear model works well, but that the intercept is not significant

Interpretations

Interpretations of coefficients is exactly the same as before:

Slope ($\mathbf{b_1}$): how much the prediction for y ($\hat{y}$) changes when we change the X variable

Intercept ($\mathbf{b_0}$): our prediction for y ($\hat{y}$) when X = 0

MPG example: $\widehat{city} = b_0 + b_1 \times (hwy) = 0.84 + 0.68 \times (hwy)$

Slope:

- when we change the hwy mpg of a vehicle by 1, the predicted city mpg changes by 0.68

Intercept:

- when the highway mpg of a vehicle is 0, the predicted city mpg is 0.84

Key Takeaways

- Regression is a generalization of ANOVA
- The β coefficients indicate how much a change in X impacts a change in Y
- Under the null, $H_0 : \beta = 0$
- R^2 gives an estimate of explained variance that, in the case of regression with a categorical variable, is identical to the sum of between-group variability
- Likewise, the residuals correspond to the total within-group variability