

# More on Probability

Types, Distributions, Conditioning, Independence

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# Review – Key Terms

**Probability:** number between 0 and 1 representing likelihood of an event

**Sample Space:** the set of all possible outcomes of a random process

**Union:** when A or B can happen

**Intersection:** when A and B both happen

**Disjoint:** when events A and B *cannot* both happen

## General Addition Rule

- ▶  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- ▶ Special: A and B are disjoint  $\rightarrow P(A \text{ or } B) = P(A) + P(B)$

## Complement Rule

- ▶  $P(\text{not } A) = P(A^C) = 1 - P(A)$

# Today's Outline

We will keep working with probability

- ▶ types of probability
- ▶ law of large numbers
- ▶ probability distribution
- ▶ conditioning
- ▶ independence / association

# Types of Probability

## **Subjective Probability:**

- ▶ How likely an event is to happen based on someone's personal belief / experience / feelings
- ▶ Most likely different answers from different people
- ▶ Ex: prob. of a sports team winning their next game?

# Types of Probability

## Theoretical Probability:

- ▶ How likely an event is to happen based on formulas or assumptions about the event
- ▶ Common assumption: events are equally likely to happen
  - ▶ coin flips
  - ▶ dice rolling

Another example: Suppose there are 20 marbles in a bag. 2 marbles are red, 6 are blue, and 12 are green. What is the probability of pulling a blue marble?

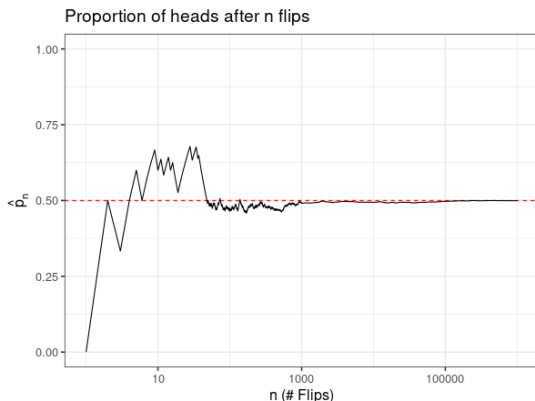
## **Empirical Probability:**

- ▶ How likely an event is to happen based on collected data
- ▶ Sometimes we estimate the probability with data in the form of a table
- ▶ Ex: flip a coin 1000 times and find the 'empirical' probability of getting a Heads

# Law of Large Numbers

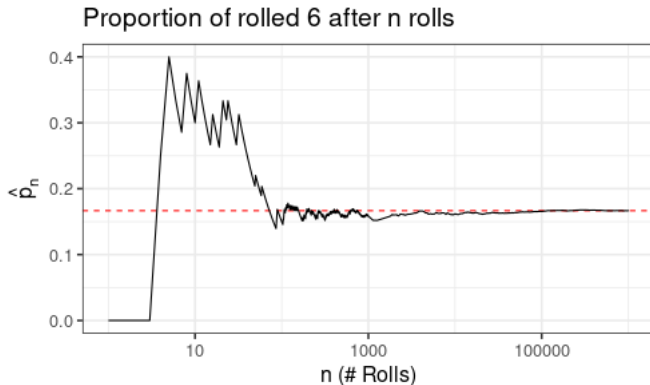
If you repeat trials a whole bunch (and they don't affect each other) then the empirical probability will converge to the "true" probability

Example: Proportion of coin flips resulting in heads after a number of flips



# Law of Large Numbers

LLN works for things other than 50/50 probabilities. For example, we saw  $P(\text{rolling a 6}) = \frac{1}{6} \approx .17 = 17\%$



As more observations are collected ( $n$  increases), the size of fluctuations of  $\hat{p}_n$  (the empirical probability) around  $p$  (the true probability) shrinks.

# Probability Distributions

Consider the scenario where we are rolling two dice and adding up the result. Since there are many different outcomes, it may be helpful to have a nice way to display those results and their probabilities.

A **probability distribution** represents each of the *disjoint* outcomes of a random process and their associated probabilities

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

- ▶ What values?
- ▶ How frequent?

# Probability Distributions

A **probability distribution** represents each of the *disjoint* outcomes of a random process and their associated probabilities

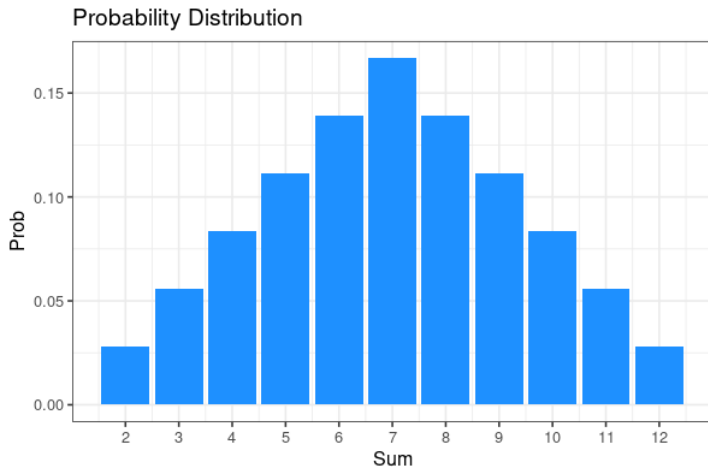
Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

For a probability distribution to be valid, the following must be true:

1. The outcomes are disjoint
2. Every probability is between 0 and 1
3. The sum of all probabilities must equal 1

# Probability Distributions

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



# Motivating Example

Consider rolling a six-sided dice with the following events:

$$A = \{2, 4\}, \quad B = \{1, 4, 6\}$$

- ▶ What is  $P(A)$ ?
- ▶ What is  $P(B)$ ?
- ▶ What is  $P(A \text{ and } B)$ ?
- ▶ If you know that  $A$  has occurred, what is the probability that  $B$  also occurred?
- ▶ If you know that  $B$  has occurred, what is the probability that  $A$  has occurred?

# Independence

We say that two *random processes* (or *events*) are **independent** if the outcome of one process provides no information about the outcome of another

Examples include:

- ▶ Flipping a coin multiple times
- ▶ Rolling a red and white dice together
- ▶ Sampling different colored marbles from a jar *with* replacement

# Independence

If  $A$  and  $B$  are two different random and *independent* processes, then the probability that both  $A$  and  $B$  occur is

$$P(A \text{ and } B) = P(A) \times P(B).$$

This is known as the **Multiplication Rule**

# Conditional Probability

When two random processes are *not* independent, we say that they are *associated*, meaning that the occurrence of one event provides information related to another event.

For example, if we know that event  $B$  has occurred and we want to assess the probability that  $A$  also occurred, we are looking for *the probability of  $A$  given  $B$*  which we will denote as  $P(A|B)$

If  $A$  and  $B$  are independent, that is, if  $B$  occurring tells us nothing new about the probability of  $A$ , then

$$P(A|B) = P(A)$$

## Conditional Probability Example

Let's go back to the heart attack study data. To compute a *conditional* probability, we look at the 'given' variable first before calculating

Treatment	Heart Attack		Total
	Attack	No Attack	
Placebo	189	10,845	11,034
Aspirin	104	10,933	11,037
Total	293	21,778	22,071

- ▶  $P(\text{HA}) = \frac{293}{22071} = 1.32\%$
- ▶  $P(\text{HA given Aspirin}) = \frac{104}{11037} = 0.94\%$
- ▶ Independent? (Does aspirin affect rate of heart attacks?)

# Conditional Probability

Let's develop a formula for conditional probabilities without a table.

	Heart Attack		
Treatment	Attack	No Attack	Total
Placebo	189	10,845	11,034
Aspirin	104	10,933	11,037
Total	293	21,778	22,071

►  $P(\text{HA given Aspirin}) = \frac{104}{11037} = 0.94\%$

What do the parts of the fraction for  $P(\text{HA given Aspirin})$  correspond to?

$$P(\text{HA given Aspirin}) = \frac{104/22071}{11037/22071} = \frac{P(\text{HA and Aspirin})}{P(\text{Aspirin})}$$

- The conditional probability takes the prob. of both Heart Attack and Aspirin, and adjusts for the prob. of aspirin

# Conditional Probability

**Conditional Probability** – probability of event A occurring if event B has already happened

- ▶  $P(A \text{ if } B)$ ,  $P(A \text{ given } B)$ ,  $P(A|B)$
- ▶ we look at the 'given' variable first before calculating our probability

Conditional probabilities are defined in terms of the *joint* (intersection) probability and *marginal* probability

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

- ▶ We adjust the joint probability  $P(A \text{ and } B)$  to account for the probability of  $B$  happening in the first place

# Independence

We now have two equivalent definitions of independence.

## **Joint Probabilities** (Multiplication Rule)

If  $A$  and  $B$  are two different random and *independent* processes, then the probability that both  $A$  and  $B$  occur is

$$P(A \text{ and } B) = P(A) \times P(B).$$

**Conditional Probabilities**  $A$  and  $B$  are independent events if knowing one has occurred does not give you any info about the other occurring

- ▶  $P(A \text{ given } B) = P(A)$
- ▶  $P(B \text{ given } A) = P(B)$

# General Multiplication Rule

When two events  $A$  and  $B$  are associated (not independent) the multiplication rule for joint probabilities that we just saw does not work.

Stating again the definition of conditional probability:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Rewriting this gives us the **General Multiplication Rule**:

$$P(A \text{ and } B) = P(A|B) \times P(B)$$

Note that when  $A$  and  $B$  are independent,  $P(A|B) = P(A)$ , giving us our original **Multiplication Rule**:

$$P(A \text{ and } B) = P(A) \times P(B)$$

# Independent $\neq$ Disjoint

The concepts of *independence* and *disjoint* events are often confused with each other, but they are not the same thing.

- ▶ independent events do not affect each others occurrence, but might still happen at the same time
- ▶ disjoint events are actually *perfectly dependent*

Consider flipping two separate fair coins. The events 'getting a heads on the first coin' and 'getting a heads on the second coin' are independent, but can both occur so are not disjoint.

Consider flipping a coin just one time and the events 'Heads' and 'Tails'. These events are disjoint, but dependent.

- ▶  $P(H \text{ given } T) = 0 \neq P(H) = 0.5$
- ▶  $P(T \text{ given } H) = 0 \neq P(T) = 0.5$

# Repeated Multiplication Rule

If we have many *independent* events we can calculate their joint (union) probability by using the **Multiplication Rule** repeatedly

Consider flipping 3 coins. What is the probability they all land Heads up? Since they are independent:

$$\begin{aligned} P(\text{all heads}) &= P(\text{H on coin 1}) \times P(\text{H on coin 2}) \times P(\text{H on coin 3}) \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \end{aligned}$$

# Summary

## Probability Distributions

- ▶ lists outcomes and gives probabilities of each outcomes
- ▶ can organize this into a graphic or table

## Conditional Probabilities

- ▶ probability of A occurring if B has occurred
- ▶  $P(A \text{ if } B) = \frac{P(A \text{ and } B)}{P(B)}$

## Independence

- ▶ Knowing about one event occurring tells nothing about the other
- ▶  $P(A \text{ and } B) = P(A) \times P(B)$
- ▶  $P(A \text{ given } B) = P(A)$

## Multiplicative Rule

- ▶  $P(A \text{ and } B) = P(A \text{ if } B) \times P(B) = P(B \text{ if } A) \times P(A)$
- ▶ Special: A and B are independent  $\rightarrow P(A \text{ and } B) = P(A) \times P(B)$