

# Probability

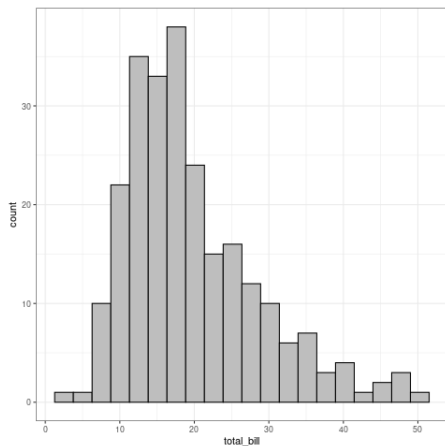
## Expectation and Variance

Grinnell College

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# Review – Describing Distributions

When we saw *distributions* earlier in the class, they were a way to represent information from a quantitative variable.



Two of the big things we talked about were center and spread.

- Measures of center we used were median or mean
- Measures of spread we used were IQR or std. dev.

# Review – Describing Distributions

**Mean** is the same thing as the **average** value of the variable.

$$\bar{x} = \frac{\sum x_i}{n}$$

- even if a distribution is skewed, the mean can still be useful
  - ▶ more on this in a sec

**Standard Deviation** is one of the measures of spread

$$s = \sqrt{\frac{1}{n-1} \sum (x_i - \bar{x})^2}$$

- interpretation: the average distance of observations from the mean
- larger value of  $s \rightarrow$  more variability or spread
- sometimes variance is used instead (variance =  $s^2$ )

# Goal for Today

We are going to apply the concept of center and spread (mean and standard deviation) to the probability distributions concept we saw on Wednesday.

# Review – Probability Distributions

A **probability distribution** represents each of the *disjoint* outcomes of a random process and their associated probabilities

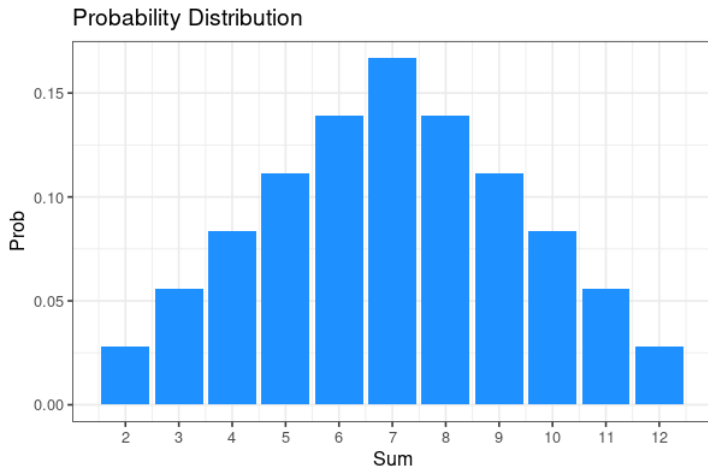
Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

For a probability distribution to be valid, the following must be true:

1. The outcomes are disjoint
2. Every probability is between 0 and 1
3. The sum of all probabilities must equal 1

# Review – Probability Distributions

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



# Random Variable

When working with a *random process* (like die rolling, or coin flipping) we can construct a quantitative variable that tells us about the outcome of that process.

Typically we will label a random variable with capital letters to distinguish it from data variables in our data sets.

**Example 1:** rolling a six-sided die

- $X$  = result of die roll (can be 1,2,3,4,5 or 6)
- $P(X=6) = \frac{1}{6}$

**Example 2:** coin flip

- $Y = 1$  if heads,  $Y = 0$  if tails
- $P(Y=1) = P(H) = P(T) = P(Y = 0) = 0.5$

# Expectation

When talking about the center of a probability distribution, most often the *mean* is used (even if the distribution is skewed). We will use the term **Expected Value** to denote that this is an average for a *random variable*.

To compute the **Expected Value**, we need to use the outcomes and account for how likely they are to come up. The *expected value* of a random variable  $X$  is denoted  $E(X)$

**Formula:**

$$E(X) = \sum x_i p_i$$

- $x_i$  represent the value of outcome  $i$
- $p_i$  is the probability associated with outcome  $i$



## Expectation – Example

Let's look at the probability distribution for rolling one 6-sided die.

Die Result	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Let  $X$  = result of roll, then

$$\begin{aligned} E(X) &= \sum x_i p_i \\ &= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) \\ &= \frac{21}{6} = 3.5 \end{aligned}$$

- Interpretation: the *expected* outcome is 3.5
- Interpretation: If you roll many 6-sided dice and compute the average, you can expect a value close to 3.5

# Expectation for Multiple RV's

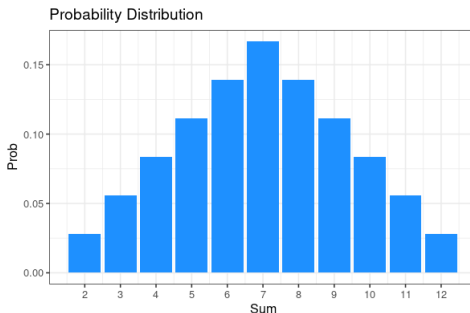
Let's return to the 2 die rolling example. When computing the expected value of the sum, it is time consuming because of how many outcomes there are. There is a simpler rule.

## Expectation of Sums

Let  $X$  and  $Y$  be two random variables. Their sum  $X + Y$  has expected value  $E(X + Y) = E(X) + E(Y)$

This works for more than 2 RV's too. The rule makes finding an average of lots of processes very easy

# Sum of Dice Example



For rolling 2 dice and adding them up, visually we can see the expected value (mean) should be 7. Let's use the result from the last slide

- Let  $X$  = result of die 1
- Let  $Y$  = result of die 2
- Let  $Z$  = sum of two dice =  $X + Y$
- Then  $E(Z) = E(X) + E(Y) = 3.5 + 3.5 = 7$

# Variance

We may also want to talk about the *variability* of a process. It's nice to know the average of a d6 is 3.5, but how much variability can I expect when I roll it?

Working with **variance** ( $=s^2$ ) is usually easier than std.dev. directly, although interpretations with std.dev. are easier

- variance of a random variable is denoted  $\text{Var}(X)$

**Formula:** Let  $\mu = E(X)$ .

$$\begin{aligned}\text{Var}(X) &= E[(X - \mu)^2] = E(X^2) - \mu^2 \\ &= (\sum x_i^2 p_i) - \mu^2\end{aligned}$$

- variance = expected squared deviation from the mean
- we won't do much calculation of Variance directly
- convert to std.dev. to do interpretations

# Variance Rules

There are rules for variances similar to expected values when working with multiple random variables (you won't be tested on these, but *simple* HW problems may show up)

- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$  if  $X$  and  $Y$  are independent
- $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y)$  if  $X$  and  $Y$  are independent
- $\text{Var}(cX) = c^2\text{Var}(X)$

## Variance Example

Let's go back to the 6-sided die rolling example.

Die Result	1	2	3	4	5	6
Probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Let  $X$  = result of roll, then

$$\begin{aligned} \text{Var}(X) &= \left( \sum x_i^2 p_i \right) - \mu^2 \\ &= \left[ 1^2 \left( \frac{1}{6} \right) + 2^2 \left( \frac{1}{6} \right) + 3^2 \left( \frac{1}{6} \right) + 4^2 \left( \frac{1}{6} \right) + 5^2 \left( \frac{1}{6} \right) + 6^2 \left( \frac{1}{6} \right) \right] - 3.5^2 \\ &= \frac{91}{6} - 3.5^2 = \frac{105}{36} \approx 2.92 \end{aligned}$$

- variance = 2.92  $\rightarrow$  s.d. =  $\sqrt{2.92} = 1.71$
- interpretation: a d6 result is 1.71 away from the mean, on average