Probability 2

Distributions, Conditioning, Independence

Grinnell College

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Review – Key Terms

Probability: number between 0 and 1 representing likelihood of an event
Sample Space: the set of all possible outcomes of a random process
Union: when A or B can happen
Intersection: when A and B both happen
Disjoint: when events A and B *cannot* both happen

General Addition Rule

- P(A or B) = P(A) + P(B) - P(A and B)
- Special: A and B are disjoint $\rightarrow$ P(A or B) = P(A) + P(B)

Complement Rule

- P(not A) = P(A^C) = 1 - P(A)

Today's Outline

We will keep working with probability

- types of probability
- law of large numbers
- probability distribution
- conditioning
- independence / association

Types of Probability

Subjective Probability:

- How likely an event is to happen based on someone's personal belief / experience / feelings
- Most likely different answers from different people
- Ex: prob. of a sports team winning their next game?

Types of Probability

Theoretical Probability:

- How likely an event is to happen based on formulas or assumptions about the event
- Common assumption: events are equally likely to happen
  - coin flips
  - dice rolling

Another example: Suppose there are 20 marbles in a bag. 2 marbles are red, 6 are blue, and 12 are green. What is the probability of pulling a blue marble?

Types of Probability

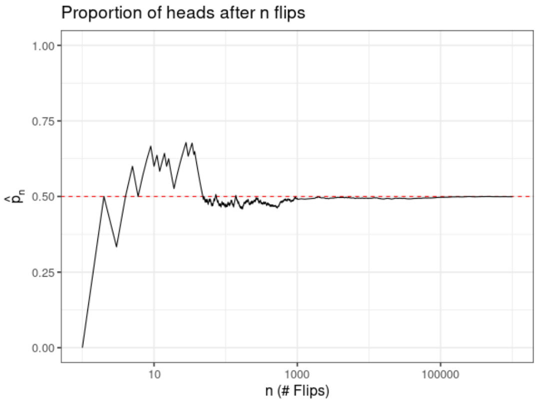
Empirical Probability:

- How likely an event is to happen based on collected data
- Sometimes we estimate the probability with data in the form of a table
- Ex: flip a coin 1000 times and find the 'empirical' probability of getting a Heads

Law of Large Numbers

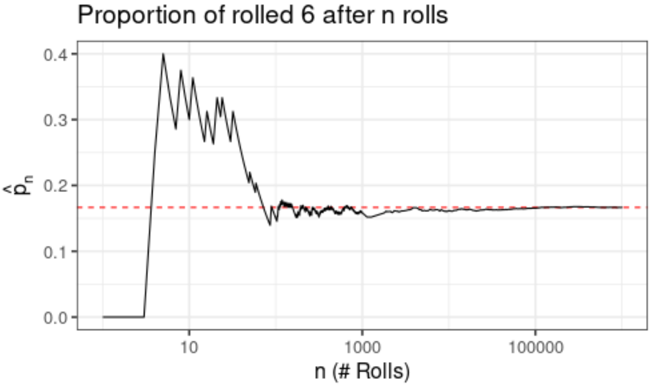
If you repeat trials a whole bunch (and they don't affect each other) then the empirical probability will converge to the "true" probability

Example: Proportion of coin flips resulting in heads after a number of flips



Law of Large Numbers

LLN works for things other than 50/50 probabilities. For example, we saw P(rolling a 6) $= \frac{1}{6} \approx .17 = 17\%$



As more observations are collected (n increases), the size of fluctuations of $\hat{p}_n$ (the empirical probability) around p (the true probability) shrinks.

Probability Distributions

Consider the scenario where we are rolling two dice and adding up the result. Since there are many different outcomes, it may be helpful to have a nice way to display those results and their probabilities.

A **probability distribution** represents each of the *disjoint* outcomes of a random process and their associated probabilities

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

- What values?
- How frequent?

A **probability distribution** represents each of the *disjoint* outcomes of a random process and their associated probabilities

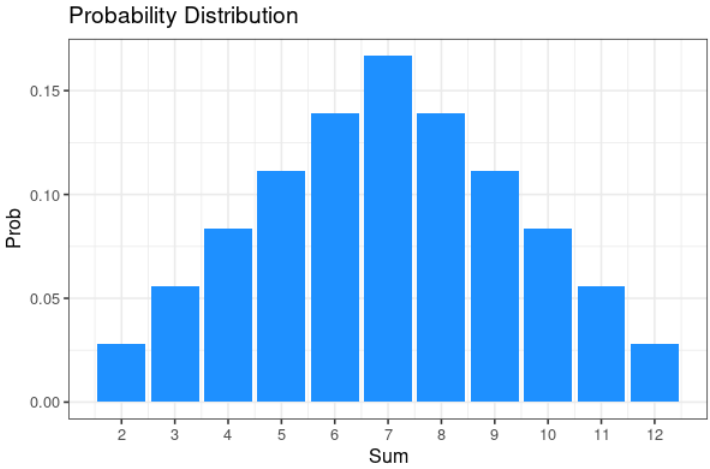
Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

For a probability distribution to be valid, the following must be true:

1. The outcomes are disjoint
2. Every probability is between 0 and 1
3. The sum of all probabilities must equal 1

Probability Distributions

Dice Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$



Motivating Example

Consider rolling a six-sided dice with the following events:

$$A = \{2, 4\}, \qquad B = \{1, 4, 6\}$$

- What is $P(A)$?
- What is $P(B)$?
- What is $P(A \text{ and } B)$?
- If you know that A has occured, what is the probability that B also occured?
- If you know that B has occured, what is the probability that A has occured?

Independence

We say that two *random processes* (or *events*) are **independent** if the outcome of one process provides no information about the outcome of another

Examples include:

- Flipping a coin multiple times
- Rolling a red and white dice together
- Sampling different colored marbles from a jar *with* replacement

Independence

If A and B are two different random and *independent* processes, then the probability that both A and B occur is

$$P(A \text{ and } B) = P(A) \times P(B).$$

This is known as the **Multiplication Rule**

Conditional Probability

When two random processes are *not* independent, we say that they are *associated*, meaning that the occurrence of one event provides information related to another event.

For example, if we know that event B has occurred and we want to assess the probability that A also occurred, we are looking for *the probability of A given B* which we will denote as $P(A|B)$

If A and B are independent, that is, if B occurring tells us nothing new about the probability of A, then

$$P(A|B) = P(A)$$

Conditional Probability Example

Let's go back to the heart attack study data. To compute a *conditional* probability, we look at the 'given' variable first before calculating

	Heart Attack		
Treatment	Attack	No Attack	Total
Placebo	189	10,845	11,034
Aspirin	104	10,933	11,037
Total	293	21,778	22,071

- $\text{P(HA)} = \frac{293}{22071} = 1.32\%$
- $\text{P(HA } \textit{given} \text{ Aspirin)} = \frac{104}{11037} = 0.94\%$
- Independent? (Does aspirin affect rate of heart attacks?)

Conditional Probability

Let's develop a formula for conditional probabilities without a table.

	Heart Attack		
Treatment	Attack	No Attack	Total
Placebo	189	10,845	11,034
Aspirin	104	10,933	11,037
Total	293	21,778	22,071

- P(HA *given* Aspirin) $= \frac{104}{11037} = 0.94\%$

What do the parts of the fraction for P(HA *given* Aspirin) correspond to?

$$\text{P(HA } \textit{given} \text{ Aspirin)} = \frac{104/22071}{11037/22071} = \frac{\text{P(HA and Aspirin)}}{\text{P(Aspirin)}}$$

- The conditional probability takes the prob. of both Heart Attack and Aspirin, and adjusts for the prob. of aspirin

Conditional Probability

Conditional Probability – probability of event A occuring if event B has already happened

- P(A if B), P(A given B), P(A|B)
- we look at the 'given' variable first before calculating our probability

Conditional probabilites are defined in terms of the *joint* (intersection) probability and *marginal* probability

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

- We adjust the joint probability P(A and B) to account for the probability of B happening in the first place

Independence

We now have two equivalent definitions of independence.

Joint Probabilities (Multiplication Rule)
A and B are independent events if the probability that both A and B occur is

$$P(A \text{ and } B) = P(A) \times P(B).$$

Conditional Probabilities (definition)
A and B are independent events if knowing one has occurred does not give you any info about the other occurring

- $\text{P}(A \text{ given } B) = \text{P}(A)$
- $\text{P}(B \text{ given } A) = \text{P}(B)$

General Multiplication Rule

When two events A and B are associated (not independent) the multiplication rule for joint probabilities that we just saw does not work.

Stating again the definition of conditional probability:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

Rewriting this gives us the **General Multipliction Rule**:

$$P(A \text{ and } B) = P(A|B) \times P(B)$$

Note that when A and B are independent, $P(A|B) = P(A)$, giving us our original **Multipliction Rule**:

$$P(A \text{ and } B) = P(A) \times P(B)$$

Independent $\neq$ Disjoint

The concepts of *independence* and *disjoint* events are often confused with each other, but they are not the same thing.

- independent events do not affect each others occurrence, but might still happen at the same time
- disjoint events are actually *perfectly dependent*

Independent $\neq$ Disjoint

Example 1
Consider flipping two separate fair coins. The events 'getting a heads on the first coin' and 'getting a heads on the second coin' are independent, but can both occur so are not disjoint.

Example 2
Consider flipping a coin just one time and the events 'Heads' and 'Tails'. These events are disjoint, but dependent.

- P(H given T) $= 0 \neq$ P(H) $= 0.5$
- P(T given H) $= 0 \neq$ P(T) $= 0.5$

Repeated Multiplication Rule

If we have many *independent* events we can calculate their joint (union) probability by using the **Multiplication Rule** repeatedly

Consider flipping 3 coins. What is the probability they all land Heads up? Since they are independent:

$$\begin{aligned} P(\text{all heads}) &= P(\text{H on coin 1}) \times P(\text{H on coin 2}) \times P(\text{H on coin 3}) \\ &= \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8} \end{aligned}$$

Summary

Probability Distributions

- lists outcomes and gives probabilities of each outcomes
- can organize this into a graphic or table

Conditional Probabilities

- probability of A occurring if B has occurred
- $\text{P(A if B)} = \frac{\textbf{P(A and B)}}{\textbf{P(B)}}$

Independence

- Knowing about one event occurring tells nothing about the other
- $\text{P(A and B)} = \text{P(A)} \times \text{P(B)}$
- $\text{P(A given B)} = \text{P(A)}$

Multiplicative Rule

- $\text{P(A and B)} = \text{P(A if B)}\times\text{P(B)} = \text{P(B if A)}\times\text{P(A)}$
- Special: A and B are independent $\rightarrow$ $\text{P(A and B)} = \text{P(A)}\times\text{P(B)}$