Probability 3

Odds and Risk

Grinnell College

Fall 2025

Outline for Today

- Introduce odds (another likelihood comparison)
- Odds ratios
- Relative Risk

Odds and Probability

When dealing with a *binary* event, we often speak in terms of **odds**, a *ratio* of "number of successes" to "number of failures"

$$\#\text{ success} : \#\text{ failure}$$

This is distinct from the idea of **probabilities**, which give a ratio of the "number of successes" to the number of possible outcomes

$$\frac{\#\text{ success}}{\#\text{ total outcomes}} = \frac{\#\text{ success}}{\#\text{ success} + \#\text{ failure}}$$

Odds

Suppose we have a 6-sided die, and we are interested in rolls that land on either 1 or 2 (note how we have turned six distinct outcomes into two "events").

$$\text{Die} = \{1, 2, 3, 4, 5, 6\}$$

- The *probability* of rolling a 1 or 2 is $1/3$
  1. There are 6 possible outcomes
  2. There are 2 possible successes
  3. Probability is $2/6 = 1/3$
- The *odds* of rolling a 1 or 2 are 2:4 (or 1:2)
  1. There are 2 possible successes
  2. There are 4 possible failures
  3. The odds of success are 2:4 (or 1:2)

Interpreting Odds

Interpretation
If the odds of success for an event is X:Y, if we repeat the random process many times then for every X number of "successes" we will have about Y number of "failures"

Hint: It is usually easiest to reduce odds down to something that like 1:X or X:1.

Example: 6-sided die

- odds of rolling either a 1 or a 2 were 2:4
- for every 2 successes (rolling 1 or 2) we expect 4 failures (rolling 3+)

Example: 6-sided die (simplified)

- odds of rolling either a 1 or a 2 are 1:2 (factored out the 2)
- for every success (rolling 1 or 2) we expect 2 failures (rolling 3+)

Empirical Example

1988 Harvard Medical School clinical trial with middle-aged male physicians

	Heart Attack		
Treatment	Attack	No Attack	Total
Placebo	189	10,845	11,034
Aspirin	104	10,933	11,037
Total	293	21,778	22,071

Probability of a heart attack (in general): $\text{P(HA)} = \frac{293}{22071} = .0132$.

- 1.32% of randomly chosen physicians had a heart attack

Odds of a heart attack: 293:21778 ($\approx 1:74$)

Interpretation:

- For every 293 heart attacks there are 21778 instances of not having a heart attack (not that helpful)
- For every heart attack there are 74 instances of not having a heart attack (more useful)

Empirical Example

	Heart Attack		
Treatment	Attack	No Attack	Total
Placebo	189	10,845	11,034
Aspirin	104	10,933	11,037
Total	293	21,778	22,071

We saw that we could compare heart attack rates between the placebo and aspiring group

- P(HA given Aspirin) $= \frac{104}{11037} = .009$
- P(HA given Placebo) $= \frac{189}{11034} = .017$
- heart attack probability is nearly half as likely for aspirin group (good!) $\rightarrow$ not independent

Empirical Example

	Heart Attack		
Treatment	Attack	No Attack	Total
Placebo	189	10,845	11,034
Aspirin	104	10,933	11,037
Total	293	21,778	22,071

We can do the same thing with odds

- heart attack odds for aspirin group: 104:10933 ($\approx$ 1:105)
- heart attack odds for placebo group: 189:10845 ($\approx$ 1:57)
- odds of heart attack for aspirin group is less than placebo group (good!) $\rightarrow$ not independent

It is harder to directly compare the odds here as a *bigger* value on the right hand side means the event is *less* likely...

Odds Examples

When working with odds, we need to keep track of the order of successes and failures and also the order of the groups in our table

	Event	Non-event
Group 1	A	B
Group 2	C	D

- The odds of an event for group 1 are A:B (or A/B)
- The odds of an event for group 2 are C:D (or C/D)

The **odds ratio** for these groups is then the ratio of their odds:

$$OR = \frac{A:B}{C:D} = \frac{A/B}{C/D} = \frac{A \times D}{B \times C}$$

Why Ratios?

Situation 1:

	Success	Failure
Group 1	6	2
Group 2	3	2

- Group 1 odds: (6:2 $\rightarrow$ 3:1), Group 2 odds: (3:2 $\rightarrow$ 1.5:1)
- OR $= \frac{6:2}{3:2} = \frac{6/2}{3/2} = 2$
- Odds of success are twice as likely for group 1

Situation 2:

	Success	Failure
Group 1	103	2
Group 2	100	2

- Group 1 odds: (103:2 $\rightarrow$ 51.5:1), Group 2 odds: (100:2 $\rightarrow$ 50:1)
- OR $= \frac{103/2}{100/2} = 1.03$
- Odds of success are basically the same for both groups

Event vs Non-Event

Which column is our "Event" or "Success" changes how we report results

Case 1:

	Survive	Death
Treatment	12	6
Placebo	5	10

- odds of survival for treatment group are 12:6 $\rightarrow$ 2:1
- surviving is twice as likely as dying for the treatment group

Case 2:

	Death	Survive
Treatment	6	12
Placebo	10	5

- odds of death for treatment group are 6:12 $\rightarrow$ 1:2
- dying is half as likely as surviving for the treatment group

Group Order

Order of groups matters when using the odds-ratio formula

Case 1:

	Survive	Death
Treatment	12	6
Placebo	5	10

- OR $= \frac{12/6}{5/10} = \frac{12*10}{5*6} = \frac{120}{30} = 4$
- odds of survival for treatment group are twice as high than for placebo

Case 2:

	Survive	Death
Placebo	5	10
Treatment	12	6

- OR $= \frac{5/10}{12/6} = \frac{5*6}{12*10} = \frac{30}{120} = \frac{1}{4} = 0.25$
- odds of survival for placebo group are 25% of treatment group

Odds and Odds Ratio Summary

Odds are another way of comparing likelihood but do not give same values as probabilities in most cases and have slightly different interpretations

- Column and row order matters for calculations and interpretations
- $OR > 1$, $OR = 1$, $OR < 1$
  - $OR = 1$ implies no association. Why?

Relative Risk

When talking about the probability of a negative outcome, some choose to call it **risk** instead

- usually refer to a conditional probability, but not always
- *risk* of heart attack for the aspirin group = P(HA given aspirin)
- *risk* of a heart attack in general = P(HA)

Just like looking at odds ratios, we can look at probability ratios. These are often called **relative risk** (because they are making comparisons)

- *relative risk* of heart attack for the aspirin group compared to placebo
- have similar interpretations to Odds-ratios
- RR $\approx 1 \rightarrow$ same probabilities
- RR $> 1 \rightarrow$ risk higher for group 1 than for group 2
- RR $< 1 \rightarrow$ risk lower for group 1 than for group 2

Relative Risk

Just like with OR, order of groups matters for RR (column order less important because we only end up using one column + total column)

	Heart Attack		
Treatment	Attack	No Attack	Total
Placebo	189	10,845	11,034
Aspirin	104	10,933	11,037
Total	293	21,778	22,071

relative risk of heart attack for placebo:

$$\frac{\textbf{P(HA if Placebo)}}{\textbf{P(HA if Aspirin)}} = \frac{189/11034}{104/11037} = \frac{189*11037}{104*11034} = 1.82$$

- Prob. of heart attack for placebo group is 1.82 times that of aspirin

relative risk of heart attack for aspirin:

$$\frac{\textbf{P(HA if Aspirin)}}{\textbf{P(HA if Placebo)}} = \frac{104/11037}{189/11034} = 0.55$$

- Prob. of heart attack for aspirin group is 0.55 times that of placebo